6

SOLUTION OF FIELD EQUATIONS

Construction & application of the electromagnetic propagators

Introduction. Working in the Lorentz gauge, our problem—acquired at (373)—is to describe the *solution* of

$$\square A^\nu = \tfrac{1}{c} j^\nu \tag{434}$$

which results when

i) the source term $j^\nu(x)$ and

ii) initial & boundary conditions

are prescribed. We will have then only to construct $F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu$ to obtain descriptions of the physical fields $\boldsymbol{E}(x)$ and $\boldsymbol{B}(x)$ that arise under the conditions specified. Our problem is made tractable by two circumstances:

- Equations (434) are *uncoupled* (though the j^μ are constrained by charge conservation to satisfy $\partial_\mu j^\mu = 0$ and the A^μ to satisfy the Lorentz gauge condition $\partial_\mu A^\mu = 0$). This means that it is suffient to study the generic equation

$$\square \phi(x) = \rho(x) \tag{435}$$

- Equations (434) are *linear*. This means that we can employ *Green's technique*; *i.e.*, that (recall the discussion on pages 16–17) we can undertake to solve (435) by weighted superposition of the solutions of

$$\square \phi(x) = \delta(x) \tag{436.1}$$

$$\square \phi(x) = 0 \tag{436.2}$$

We anticipate on these grounds that the solution of (435) can be developed

$$\phi(x) = \phi_0(x) + \int G(x-y)\rho(y)\,d^4y \tag{437}$$

$$= \left\{\begin{array}{l}\text{solution of the homogeneous equation (436.2)}\\ \text{into which we have folded the \textit{initial value data}}\end{array}\right\}$$

$$+ \big\{\text{particular solution of (435)}\big\}$$

and that the physical solutions of (434) admit of similar description:

$$A^\mu(x) = A_0^\mu(x) + \underbrace{\tfrac{1}{c}\int D_{\mathrm{R}}(x-y)\,j^\mu(y)\,d^4y} \tag{438.1}$$

Here $A_0^\mu(x)$ denotes the field which has evolved from any initially present ambient field, and

$$\equiv A_{\mathrm{R}}^\mu(x) \tag{438.2}$$

denotes the field generated by past source activity (the subscript $_{\mathrm{R}}$ stands for "retarded").

We look first to the detailed substance of the preceding rough remarks, and in subsequent sections to a graded sequence of illustrative applications.

1. Green's function techniques in classical electrodynamics: construction of the propagators. I start with remarks that—though they may seem at first to be in mathematical left field—will place us in position to say powerful things about the source-independent term $A_0^\mu(x)$.

If in Gauss' theorem

$$\iiint_{\mathcal{R}} \boldsymbol{\nabla}\!\cdot\!\boldsymbol{A}\,d^3x = \iint_{\partial\mathcal{R}} \boldsymbol{A}\!\cdot\!\boldsymbol{d\sigma}$$

we set $\boldsymbol{A} = \varphi\boldsymbol{\nabla}\psi$ we obtain

$$\iiint_{\mathcal{R}} \big\{\varphi\nabla^2\psi + \boldsymbol{\nabla}\varphi\cdot\boldsymbol{\nabla}\psi\big\}\,d^3x = \iint_{\partial\mathcal{R}} \varphi\boldsymbol{\nabla}\psi\!\cdot\!\boldsymbol{d\sigma}$$

from which (interchange φ and ψ, subtract) follows **Green's theorem**

$$\iiint_{\mathcal{R}} \big\{\varphi\nabla^2\psi - \psi\nabla^2\varphi\big\}\,d^3x = \iint_{\partial\mathcal{R}} \big\{\varphi\boldsymbol{\nabla}\psi - \psi\boldsymbol{\nabla}\varphi\big\}\!\cdot\!\boldsymbol{d\sigma}$$

Green's theorem lies at the heart of many notable existence and uniqueness theorems. And it is quite robust: it extends to spaces of any dimension, and of non-Euclidean metric structure. In 4-dimensional spacetime it reads

$$\iiiint_{\mathcal{R}} \big\{\varphi\Box\psi - \psi\Box\varphi\big\}\,d^4x = \iiint_{\partial\mathcal{R}} \big\{\varphi\,\partial^\alpha\psi - \psi\,\partial^\alpha\varphi\big\}\,d\sigma_\alpha \tag{439}$$

To prepare for the application specifically at hand we

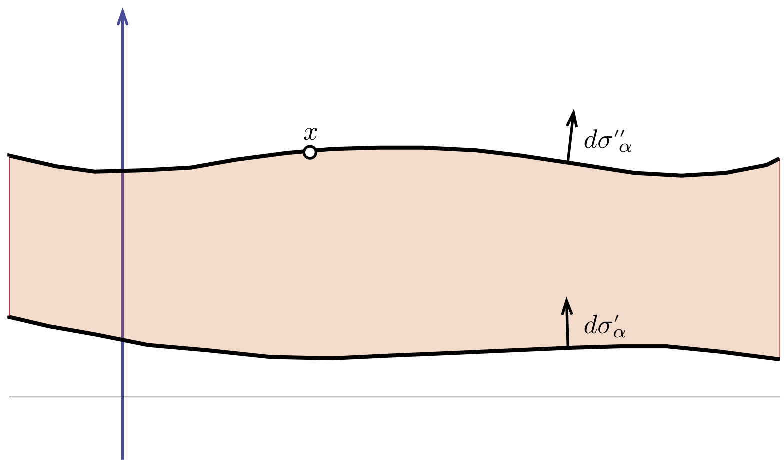

FIGURE 105: *Spacetime sandwich, bounded by surfaces the normals to which are everywhere* <u>*timelike and future directed*</u> *(the former by construction, the latter by convention). The upper and lower spacelike surfaces (or timeslices) σ' and σ'' jointly comprise the boundary $\partial\mathcal{R}$ of the region $\mathcal{R}$ which we bring in the text to a distinctive application of Green's theorem.*

1) assume both φ and ψ to satisfy (436.2): $\Box\varphi = \Box\psi = 0$

2) assume $\mathcal{R}$ to be the disk-like region bounded by the everywhere-spacelike surfaces σ' and σ'', where σ'' contains x—the field point of interest. It is out intention to spread **Cauchy data** (*i.e.*; initial data sufficient to identify/determine a solution) on σ'. . . like so much peanut butter & jelly.

3) assume the surface differentials $d\sigma'_\alpha$ and $d\sigma''_\alpha$ to be (not "outer-directed" but) <u>future-directed</u> (see the figure).

Green's equation (439), on the strength of those assumptions, becomes

$$0 = \int_{\sigma''}\big\{\varphi\,\partial^\alpha\psi - \psi\,\partial^\alpha\varphi\big\}\,d\sigma''_\alpha - \int_{\sigma'}\big\{\varphi\,\partial^\alpha\psi - \psi\,\partial^\alpha\varphi\big\}\,d\sigma'_\alpha$$

or

$$\underbrace{\int_{\sigma''}\Big\{\varphi(x'')\,\partial^\alpha\psi(x'') - \psi(x'')\,\partial^\alpha\varphi(x'')\Big\}\,d\sigma''_\alpha}$$
$$= \int_{\sigma'}\Big\{\varphi(x')\,\partial^\alpha\psi(x') - \psi(x')\,\partial^\alpha\varphi(x')\Big\}\,d\sigma'_\alpha$$

Now $$= \varphi(x) \tag{440}$$

<u>if an appropriately specialized meaning is assigned to ψ</u>. If we agree to write

$\psi(x'') \equiv D_0(x''-x)$ and to interpret x as a "continuously adjustable parameter" then we achieve (440) by stipulating that

$$\left.\begin{aligned} \Box D_0(x''-x) &= 0 \\ \int_{\sigma''} f(x'')\,\partial^\alpha D_0(x''-x)\,d\sigma''_\alpha &= f(x) \quad : \quad \begin{cases}\text{all } f\text{, and}\\ \text{all timeslices } \sigma'' \text{ through } x\end{cases} \\ D_0(x''-x) &= 0 \quad : \quad x''-x \text{ spacelike} \\ D_0(0) &= 0 \end{aligned}\right\} \tag{441}$$

It is by no means obvious that such a $D_0(\bullet)$ exists, but if it did (and it does!... as will soon be established *by construction*) we would have

$$\phi_0(x) = \int_{\sigma'} \Big\{ \underbrace{\phi_0(x')}_{\text{Cauchy data}}\,\partial^\alpha D_0(x'-x) - D_0(x'-x)\,\underbrace{\partial^\alpha\phi_0(x')}_{\text{more Cauchy data}} \Big\}\,d\sigma'_\alpha \tag{442}$$

which describes $\phi(x)$ in terms of the prescribed initial data; *i.e.*, in terms of the stipulated values assumed by ϕ and $\partial\phi$ on the spacelike surface σ'. The construction of $D_0(\bullet)$ follows (as it happens) directly from that of $D_{\text{R}}(\bullet)$, so it is to the latter—simpler—problem that I now turn:

Let $\tilde\phi(k)$ and $\tilde\rho(k)$ be the Fourier transforms of $\phi(x)$ and $\rho(x)$:

$$\begin{aligned} \phi(x) &= \big(\tfrac{1}{\sqrt{2\pi}}\big)^4 \iiiint \tilde\phi(k)e^{i(k^0x^0 - \boldsymbol{k}\cdot\boldsymbol{x})}\,dk^0dk^1dk^2dk^3 \\ &\equiv \tfrac{1}{(2\pi)^2}\int \tilde\phi(k)e^{ikx}\,d^4k \qquad (443.1)\\ \tilde\phi(k) &= \tfrac{1}{(2\pi)^2}\int \phi(x)e^{-ikx}\,d^4x \qquad (443.2) \end{aligned}$$

$$\begin{aligned} \rho(x) &= \tfrac{1}{(2\pi)^2}\int \tilde\rho(k)e^{ikx}\,d^4k \qquad (443.3)\\ \tilde\rho(k) &= \tfrac{1}{(2\pi)^2}\int \rho(x)\,e^{-ikx}\,d^4x \qquad (443.4) \end{aligned}$$

The Fourier transform of $\Box\phi(x) = \rho(x)$ is *algegbraic*

$$-k^2\tilde\phi(k) = \tilde\rho(k)$$

and admits of immediate solution:[263]

$$\tilde\phi(k) \overset{\Downarrow}{=} -\tfrac{1}{k^2}\,\tfrac{1}{(2\pi)^2}\int \rho(x)\,e^{-ikx}\,d^4x$$

[263] This development is typical of the effective application of integral transform techniques to the solution of differential equations. And it illustrates why the inhomogeneous equation $\Box\phi(x) = \rho(x)$ is so much easier to discuss than its homogeneous counterpart.

Returning with this information to (443.1), we reverse the order of integration to obtain

$$\phi(x) = \int \Big\{ -\tfrac{1}{(2\pi)^4} \int k^{-2} e^{ik(x-x)}\, d^4k \Big\} \rho(x)\, d^4x \tag{444}$$

Comparison with (437) gives

$$G(x-x) = -\tfrac{1}{(2\pi)^4} \int k^{-2} e^{ik(x-x)}\, d^4k$$
$$k^2 = k_0^2 - \boldsymbol{k}\cdot\boldsymbol{k}$$

But the integrand is <u>singular on the null-cone in k-space</u>, so the integral is *meaningless until <u>assigned</u> a meaning*. To that end, we write

$$= -\tfrac{1}{(2\pi)^3} \iiint e^{-i\boldsymbol{k}\cdot(\boldsymbol{x}-\boldsymbol{x})} \Big\{ \tfrac{1}{2\pi} \int_{-\infty}^{+\infty} \frac{1}{k_0^2 - \boldsymbol{k}\cdot\boldsymbol{k}} e^{ik_0(x^0 - x^0)}\, dk_0 \Big\}\, d^3k \tag{445}$$

which serves to localize the pathology at a pair of *points*: $k_0 = \pm\sqrt{\boldsymbol{k}\cdot\boldsymbol{k}}$. Next we resort to some standard trickery: we complexify k_0, reinterpret $\int_{-\infty}^{+\infty}$ as a contour integral $\oint$, and circumvent the simple poles at $k_0 = \pm\sqrt{\boldsymbol{k}\cdot\boldsymbol{k}}$ by contour deformation. Equation (445) is replaced thus by the meaningful but contour-dependent equation

$$G_C(x-x) = -\tfrac{i}{(2\pi)^3} \iiint e^{-i\boldsymbol{k}\cdot(\boldsymbol{x}-\boldsymbol{x})} \tag{446}$$
$$\cdot \Big\{ \tfrac{1}{2\pi i} \oint_C \frac{1}{k_0^2 - \boldsymbol{k}\cdot\boldsymbol{k}} e^{ik_0(x^0 - x^0)}\, dk_0 \Big\}\, d^3k$$

where (by the "method of partial fractions")

$$\frac{1}{k_0^2 - \boldsymbol{k}\cdot\boldsymbol{k}} = \frac{1}{2k}\Big[\frac{1}{k_0 - k} - \frac{1}{k_0 + k}\Big]$$

with $k \equiv \sqrt{\boldsymbol{k}\cdot\boldsymbol{k}}$.

We have physical interest not in all possible G_C-functions (all possible contours C, of which there are only a handful of truly distinct options: see RELATIVISTIC CLASSICAL FIELDS (1973) page 167) but only in that particular G_C—denoted $D_R(x-x)$—which conforms to our conception of "retarded causal action." It is, therefore, for *physical* reasons (see below) that we take C to have the form illustrated in Figure 106. Writing $k_0 = r + is$, we have

$$e^{ik_0(x^0 - x^0)} = e^{-s(x^0 - x^0)} \cdot e^{ir(x^0 - x^0)}$$

and it becomes clear that to achieve a finite result we must have $s \to \pm\infty$ according as $x^0 \gtrless x^0$; *i.e.*, that we must *close the contour on the upper or lower half-plane according as the source point x lies in the past or the future of the field point x*. The detours around the poles (see the figure) are now dictated by the physical requirement that <u>present field physics shall be insensitive to future source activity</u>. It now follows by the residue theorem that

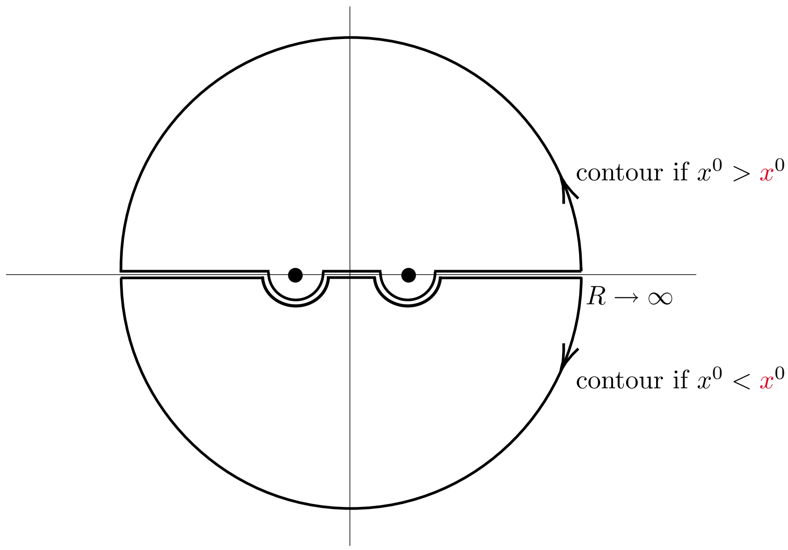

FIGURE 106: *Causal contour, inscribed on the complex k_0-plane: close on the upper half-plane if the field point x lies in the future of the source-point x ($x^0 > x^0$), and on the lower half-plane in the contrary case. The upper contour encloses the poles at $k_0 = \pm\sqrt{\boldsymbol{k}\cdot\boldsymbol{k}}$; the lower contour excludes them, so gives $\oint_C = 0$.*

$$\Big\{\text{etc.}\Big\} = \begin{cases} \dfrac{1}{2k}\Big[e^{ik(x^0 - x^0)} - e^{-ik(x^0 - x^0)}\Big] = i\,\dfrac{\sin k(x^0 - x^0)}{k} & \text{if} \quad x^0 > x^0 \\ \\ 0 & \text{if} \quad x^0 < x^0 \end{cases}$$

so

$$D_{\mathrm{R}}(x - x) = \begin{cases} \dfrac{1}{(2\pi)^3}\iiint \dfrac{\sin k(x^0 - x^0)}{k} e^{-i\boldsymbol{k}\cdot(\boldsymbol{x} - \boldsymbol{x})}\, d^3k \\ \\ 0 \end{cases} \tag{447}$$

To facilitate evaluation of the $\iiint$ we introduce spherical coordinates into $\boldsymbol{k}$-space (3-axis parallel to $\boldsymbol{x} - \boldsymbol{x}$) and (in the case $x^0 > x^0$) obtain

$$= \frac{1}{(2\pi)^3}\int_0^\infty\!\!\int_0^\pi\!\!\int_0^{2\pi} \frac{\sin k\xi^0}{k} e^{-ik\xi\cos\phi} k^2 \sin\phi \, d\theta d\phi dk$$

where $\xi^0 \equiv x^0 - x^0$ and $\xi \equiv \sqrt{(\boldsymbol{x} - \boldsymbol{x})\cdot(\boldsymbol{x} - \boldsymbol{x})} \geqslant 0$. Immediately

$$= \frac{1}{(2\pi)^2}\int_0^\infty \underbrace{2\sin\xi^0 k\cdot\frac{\sin\xi k}{\xi}}_{=\frac{1}{\xi}\Big[\cos k(\xi^0-\xi)-\cos k(\xi^0+\xi)\Big]}\,dk$$

$$= \frac{1}{(2\pi)^2}\frac{1}{\xi}\lim_{k\to\infty}\Big[\frac{\sin k(\xi^0-\xi)}{(\xi^0-\xi)}-\frac{\sin k(\xi^0+\xi)}{(\xi^0+\xi)}\Big]$$

But $\delta(x) = \frac{1}{\pi}\lim_{k\to\infty}\frac{\sin kx}{x}$ provides a standard *parameterized representation of the Dirac δ-function*,[264] so

$$= \frac{1}{(2\pi)^2}\frac{\pi}{\xi}\Big[\delta(\xi^0-\xi)-\delta(\xi^0+\xi)\Big] \tag{448}$$

The 2nd δ-function is moot when $\xi^0 > 0$ (*i.e.*, when x^0 and x^0 stand in causal sequence: $x^0 > x^0$), while according to (447) *both* terms are extinguished when $x^0 < x^0$. We come thus to the conclusion that

$$D_{\rm R}(x-x) = \begin{cases} \dfrac{1}{4\pi\xi}\delta(\xi^0-\xi) & : \quad \xi^0 > 0 \\ \\ 0 & : \quad \xi^0 < 0 \end{cases} \tag{449.1}$$

Were we to deform the contour C so as instead to favor *advanced action* (fields responsive to *future* source activity!) we would, by the same analysis, be led to

$$D_{\rm A}(x-x) = \begin{cases} 0 & : \quad \xi^0 > 0 \\ \\ \dfrac{1}{4\pi\xi}\delta(\xi^0+\xi) & : \quad \xi^0 < 0 \end{cases} \tag{449.2}$$

The *retarded* and *advanced propagators* (or Green's functions) $D_{\rm R}(\bullet)$ and $D_{\rm A}(\bullet)$ are, in an obvious sense, "natural companions." The former, according to (448), <u>vanishes except on the lightcone that extends backwards from the fieldpoint x</u>, while $D_{\rm A}(\bullet)$ vanishes except on the <u>forward</u> lightcone: see Figure 107.

What about the function $D_0(x-x)$? It has, as I will show, been sitting quitely on the right side of (448):

$$\begin{aligned} D_0(x-x) &= \frac{1}{4\pi\xi}\Big[\delta(\xi^0-\xi)-\delta(\xi^0+\xi)\Big] \quad : \quad \text{all } \xi^0 \\ &= D_{\rm R}(x-x) - D_{\rm A}(x-x) \end{aligned} \tag{450}$$

Note first that $D_0(x-x)$—thus described, and thought of as a function of x—clearly vanishes except on the lightcone that extends backward and forward

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[264] To see how the representation does its job, use *Mathematica* to `Plot` the function $\frac{\sin kx}{\pi x}$ for several values of k, and also to evaluate $\int_{-\infty}^{+\infty}\frac{\sin kx}{\pi x}\,dx$.

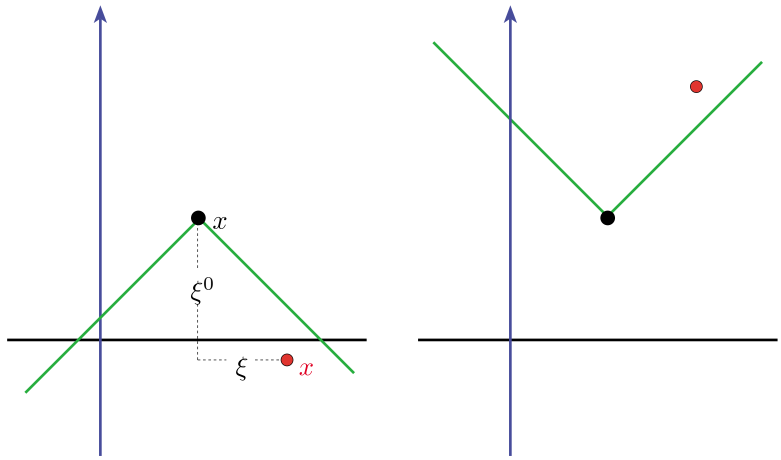

FIGURE 107: *The retarded propagator $D_{\rm R}(\bullet)$ harvests source data written onto the lightcone (shown at left) that extends backward from the fieldpoint $\bullet$. The advanced propagator $D_{\rm A}(\bullet)$ looks similarly to the forward lightcone. Source data at the $\bullet$ shown at left is actually invisible to the fieldpoint $\bullet$, since it lies interior to rather than on the backward cone (but it would become visible* if the photon had mass). *Ditto at right.*

from x, so the 3rd of the conditions (441) is clearly satisfied. Writing

$$D_0(x-x)\equiv \mathcal{D}(\xi^0,\xi)$$

we observe that $\mathcal{D}(\xi^0,\xi)$ is, by (450), an *odd* function of ξ^0, so

$$\mathcal{D}(0,\xi)=0 \quad : \quad \text{all } \xi$$

which serves to establish the 4th of the conditions (441). That $\Box D_0(x-x)=0$ (the 1st of those conditions) follows from the remarks (i) that the functions $G_{\rm C}(x-x)$ described at (446) satisfy $\Box G_{\rm C}=0$ for *every* contour C, and (ii) that $G_{\rm C}\to D_0$ if we take C to be (topologically equivalent to) the bounded contour shown in Figure 108. Finally, we observe (see again (447))that

$$\begin{aligned}\frac{\partial}{\partial x^0}D_0(x-x)&=\frac{1}{(2\pi)^3}\iiint \cos k(x^0-x^0)e^{-i\boldsymbol{k}\cdot(\boldsymbol{x}-\boldsymbol{x})}\,d^3k\\ &\downarrow\\ &=\frac{1}{(2\pi)^3}\iiint e^{-i\boldsymbol{k}\cdot(\boldsymbol{x}-\boldsymbol{x})}\,d^3k \quad \text{when}\quad x^0=x^0\end{aligned}$$

But $$=\delta(\boldsymbol{x}-\boldsymbol{x})$$

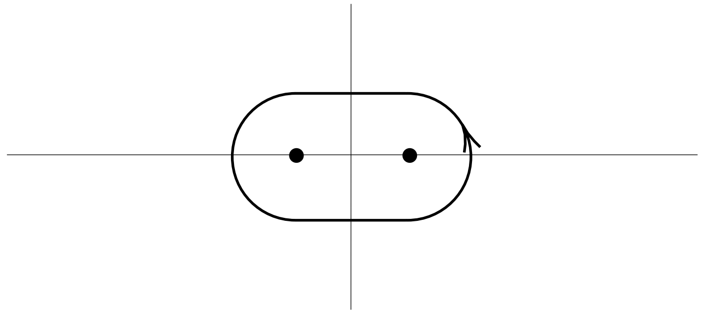

FIGURE 108: *The* *<u>bounded</u>* *contour that, when introduced into* (446), *yields the function* D_0. *The contours shown in Figure 106 have the property that they are "this or that, depending on the sign of the time," and it is because they "flip" that they give rise to a solution of the* inhomogeneous *wave equation. The contour shown above entails no such flip, so gives rise to a solution of the* homogeneous *wave equation. The point is developed in the text, and—in much great detail—in a reference cited.*

by the *Fourier integral theorem*,[265] and this expresses the upshot of the 2^{nd} of the conditions (441). Further analysis would show that the $D_0(x-{\color{red}x})$ described above is the *unique* realization of the conditions (441).

Returning with (450) to (447) we obtain

$$\begin{aligned} D_{\rm R}(x-{\color{red}x}) &= \quad\theta(\ \ x^0-{\color{red}x}^0)\cdot D_0(x-{\color{red}x}) \\ D_{\rm A}(x-{\color{red}x}) &= -\theta(-x^0+{\color{red}x}^0)\cdot D_0(x-{\color{red}x}) \end{aligned}$$

where $\theta(x)$ is the *Heaviside step function*:

$$\theta(x)=\int_{-\infty}^{x}\delta(\xi)\,d\xi=\begin{cases} 0 & x<0 \\ \frac{1}{2} & x=0 \\ 1 & x>0 \end{cases}$$

It's occurance in this context can be traced to the sign-of-the-times-dependent "contour flipping" that enters into the definitions of $D_{\rm R}(x-{\color{red}x})$ and $D_{\rm A}(x-{\color{red}x})$

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[265] The Fourier integral theorem asserts that

$$\phi(x)=\tfrac{1}{\sqrt{2\pi}}\int e^{ikx}\left\{\tfrac{1}{\sqrt{2\pi}}\int e^{-ik{\color{red}x}}\phi({\color{red}x})\,d{\color{red}x}\right\}dk$$

for "all" $\phi(x)$. Reversing the order of integration, we obtain the identity used in the text

$$\delta({\color{red}x}-x)=\tfrac{1}{2\pi}\int e^{-ik({\color{red}x}-x)}\,dk$$

which can be considered to lie at the heart of Fourier's theorem and of Fourier analysis.

(see again Figure 106) but is absent from the definition of $D_0(x - x)$.[266]

From the fact that $D_0(\bullet)$ is attached to *both sectors* of the lightcone we conclude (see again (442)) that if we know the values assumed by the free ambient field ϕ_0 and its derivatives $\partial\phi_0$ on *some* spacelike surface σ' then we know the values assumed *everywhere* by ϕ_0: the free field equations allow us both to *predict* and to *retrodict*. But the field equations do not, in general, allow us to predict *source* motion, which is typically of semi-extrinsic origin (we haven't yet decided whether to flip the light switch or not!) . . . and it is for this reason that we have—"by hand," not from mathematical (or deep physical?) necessity—inserted $D_\mathrm{R}(\bullet)$ rather than $D_\mathrm{A}(\bullet)$ into (438.1).

The preceding analysis has been somewhat "heavy." But it has yielded results—see again (438), (442), (449) & (450)—of remarkable simplicity and high plausibility. It has employed analytical methods which have in fact long been standard to several branches of "linearity-dominated" physics and engineering (though their importation into classical/quantum electrodynamics is—oddly—of relatively recent date: it was accomplished in the late 1940's and early 1950's by Julian Schwinger) . . . and which are, beneath the surface clutter, really rather pretty (Richard Crandall's "favorite stuff"). I turn now to discussion of some of the specific electrodynamical implications of the material now in hand.

2. Application: the Liénard-Wiechert potential. Let the values—values consistent with the Lorentz gauge condition—assumed by the 4-potential A^μ and its first derivatives $\partial^\alpha A^\mu$ on some everywhere-spacelike surface σ be given/prescribed. Then (see again (442): also Figure 109)

$$A^\mu(x) = \int_\sigma \Big\{ A^\mu(x)\partial^\alpha D_0(x - x) - D_0(x - x)\partial^\alpha A^\mu(x)\Big\}\, d\sigma_\alpha \tag{451}$$

describes the "evolved values" that—in forced consequence of the equations of free-field motion—are assumed by our "ambient field" at points x which lie off the "data surface" σ. Any *particular* inertial observer would in most cases find it most natural to take σ to be a time-slice, and in place of (451) to write

$$= \iiint \Big\{ A^\mu(x)\frac{\partial}{\partial x^0} D_0(x - x) - D_0(x - x)\frac{\partial}{\partial x^0} A^\mu(x)\Big\} dx^1 dx^2 dx^3$$

While every particular observer has that option (Figure 110), it must be borne in mind that *the time-slice concept is not boost invariant*: the point was illustrated in Figure 58, and is familiar as the "breakdown of non-local simultaneity." The preceding equation states explicitly how the value of $A^\mu(x)$ depends upon the initial value and initial time derivative of the field, and establishes the sense in which "launching a free electromagnetic field" is like throwing a ball.[267]

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[266] This topic is developed in unusual detail in §§3 & 4 of my "Simplified production of Dirac δ-function identities," (1997).

[267] PROBLEM 70.

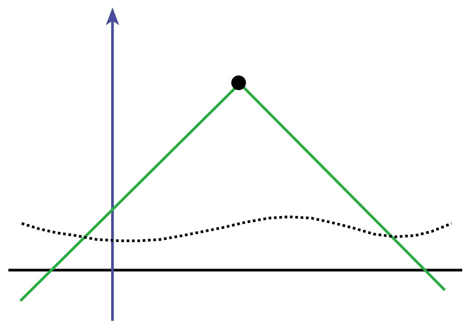

FIGURE 109: *Cauchy data is written onto the dotted surface* σ*. The function* $D_0(x - x)$ *vanishes* <u>*except on the lightcone*</u>*: it serves in* (451) *to describe how data at the intersection of* σ *with the lightcone is conflated to produce the value assumed by* A^μ *at the fieldpoint* $\bullet$*. As the temporal coordinate of* $\bullet$ *increases the intersection becomes progressively more remote, until finally it enters a region where (in typical cases) the initial data was null . . . which is to say: the ambient field at any given spatial location can be expected ultimately to die away. The die-off is reenforced by the* $(4\pi\xi)^{-1}$ *which was seen at (450) to enter into the design of* D_0*.*

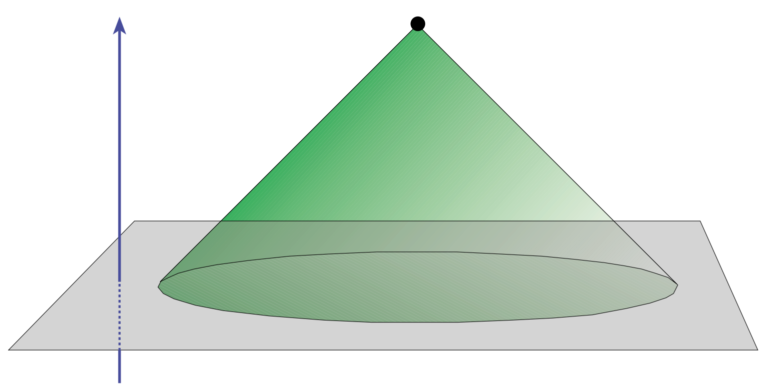

FIGURE 110: *An inertial observer has exercised his non-covariant option to deposit his Cauchy data on a time-slice. Only data at the spherical* <u>*intersect*</u> *of the lightcone and the time-slice contribute to the value assumed at* $\bullet$ *by* A^μ*, though "if the photon had mass" then data interior to the sphere would also contribute.*[2]

We turn our attention now to the component of the A^μ-field that arises from source activity, which according to (438/449) can be described

$$A^\mu(x) = \tfrac{1}{c}\int D_{\mathrm{R}}(x - x)\,j^\mu(x)\,d^4x \tag{452.1}$$

$$D_{\mathrm{R}}(x-x) = \begin{cases} \dfrac{1}{4\pi R}\delta(cT - R) & : \quad T > 0 \\ 0 & : \quad T < 0 \end{cases} \tag{452.2}$$

with $cT \equiv x^0 - x^0$ and $R \equiv |\boldsymbol{x} - \boldsymbol{x}|$. We can therefore state that the value assumed by A^μ at the field point x arises (by superposition) entirely from the source activity sampled by the lightcone which extends backward from x. In an effort to expose more clearly the meaning of this result we consider $j^\mu(x)$ to arise from a solitary point charge e in arbitrarily prescribed motion: we assume, in other words, that $j^\mu(x)$ can be described (see again (323))

$$j^\mu(x) = ec\int_{-\infty}^{+\infty} u^\mu(\tau)\delta(x - x(\tau))\,d\tau$$

Immediately

$$\begin{aligned}
A^\mu_{\mathrm{R}}(x) &= e\int D_{\mathrm{R}}(x-x)\int_{-\infty}^{+\infty} u^\mu(\tau)\delta(x - x(\tau))\,d\tau d^4x \\
&= e\int_{-\infty}^{+\infty}\int u^\mu(\tau)D_{\mathrm{R}}(x-x)\delta(x-x(\tau))\,d^4x d\tau \\
&= e\int_{-\infty}^{+\infty} u^\mu(\tau)D_{\mathrm{R}}(x - x(\tau))\,d\tau \\
&= \frac{e}{4\pi}\int_{-\infty}^{+\infty} u^\mu(t)\frac{1}{R(\tau)}\delta(G(\tau))\,d\tau \\
&\qquad\qquad G(\tau) \equiv cT(\tau) - R(\tau)
\end{aligned}$$

An elementary change-of-variables argument[268] leads to the important general conclusion that

$$\delta(g(x)) = \sum_\alpha \frac{1}{|g'(x_\alpha)|}\delta(x - x_\alpha) \tag{453}$$

where $g'(x_\alpha) \equiv \frac{d}{dx}g(x)$ and where (see Figure 111) the x_α locate the zeros of $g(x)$. It follows by way of application to the problem at hand that

$$= \frac{e}{4\pi}\int_{-\infty}^{+\infty} u^\mu(t)\frac{1}{R(\tau)}\frac{1}{|G'(\tau_0)|}\delta(\tau - \tau_0)\,d\tau \tag{454}$$

where τ_0 is the proper time at which $x(\tau)$ punctures the backward lightcone, and where $G' \equiv \frac{d}{d\tau}G$. If t_0, $\boldsymbol{x}_0$ and $\boldsymbol{v}_0$ refer to the source-particle at the instant of puncture, then we have (borrowing a trick from page 192)

[268] See ELECTRODYNAMICS (1972), page 304. An alternative argument—that makes transparent the origin of the perplexing absolute value bars—can be found in the little paper cited just above.[264]

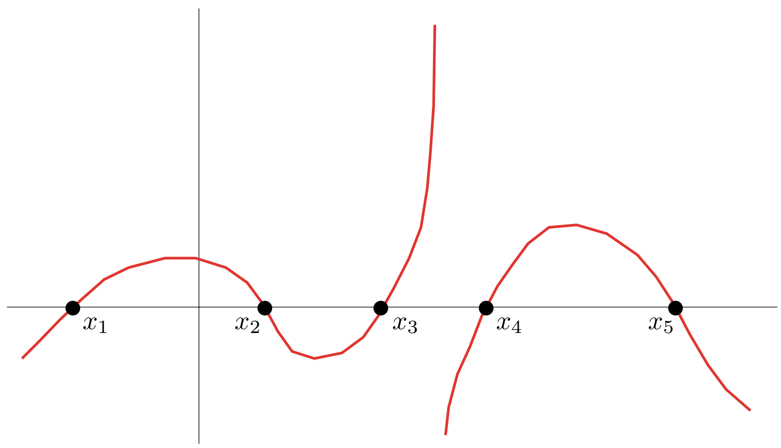

FIGURE 111: *Zeros x_α of a function $g(x)$. The numbers x_α and $g(x_\alpha)$ enter into the formulation of the important identity* (453).

$$\begin{aligned} G'(\tau_0) &= \gamma_0 \tfrac{d}{dt_0}\Big\{c(t-t_0) - \sqrt{\boldsymbol{R}\cdot\boldsymbol{R}}\Big\} \quad \text{with} \quad \boldsymbol{R} \equiv \boldsymbol{x} - \boldsymbol{x}_0 \\ &= \gamma_0\big(-c + \hat{\boldsymbol{R}}\cdot\boldsymbol{v}_0\big) \\ &= -c\gamma_0(1-\beta_\parallel)_0 \end{aligned} \tag{455}$$

$$\beta_\parallel \equiv \tfrac{1}{c}\hat{\boldsymbol{R}}\cdot\boldsymbol{v} \equiv \begin{cases} \text{magnitude of the component} \\ \text{of } \boldsymbol{\beta} \text{ that is parallel to } \boldsymbol{R} \end{cases}$$

Returning with (455) to (454) we obtain[269]

$$A^\mu_{\text{R}}(x) = \tfrac{e}{4\pi}\Big[\frac{1}{c\gamma(1-\beta_\parallel)R}u^\mu\Big]_0 \tag{456.1}$$

$\uparrow$ signifies "evaluation at the puncture point"

which—recall $A = \begin{pmatrix} \varphi \\ \boldsymbol{A} \end{pmatrix}$ and $u = \gamma\begin{pmatrix} c \\ \boldsymbol{v} \end{pmatrix}$—can also be formulated

$$\left.\begin{aligned} \varphi_{\text{R}}(x) &= \tfrac{e}{4\pi}\Big[\frac{1}{(1-\beta_\parallel)R}\Big]_0 \\ \boldsymbol{A}_{\text{R}}(x) &= \tfrac{e}{4\pi}\Big[\frac{1}{(1-\beta_\parallel)R}\boldsymbol{\beta}\Big]_0 \end{aligned}\right\} \tag{456.2}$$

Equations (456)—which are, in view of the complexity of the argument from which they derive, remarkably simple, and which describe the potential

[269] Beware! The R on the left is intended to signify "retarded," while on the right R means "length of $\boldsymbol{R}$."

fields generated by the retarded action of a moving point charge—were first obtained by A. Liénard (1898) and E. Wiechart (1900), and describe what are universally known as the **Liénard-Wiechart potentials**. The "retarded potential" idea was apparently original to B. Riemann (1859), and the essence of (452) can reportedly be found in work (1867) of Ludwig Lorenz (who, as previously remarked, is to be distinguished from H. A. Lorentz). The work of Riemann and of Lorenz was known to Maxwell, but one gets the impression (see *Treatise on Electricity & Magnetism*, §§805 and 861–end) that Maxwell was not much impressed. Which—though historically explicable—is too bad, for equations (456) are, as will emerge, *fundamental to the theory of radiative processes*.

The "advanced analogs" of (456) can be obtained by reversing the signs of all $\beta_\parallel$-terms and evaluating $\big[$etc.$\big]$ at the future puncture point.

The Liénard-Wiechart potential (456.2) gives back the familiar Coulomb potential

$$\begin{aligned}\varphi(x) &= \frac{e}{4\pi R}\\ \boldsymbol{A}(x) &= \mathbf{0}\end{aligned}$$

when the source is at rest (see the figure), and the "retarded evaluation" idea

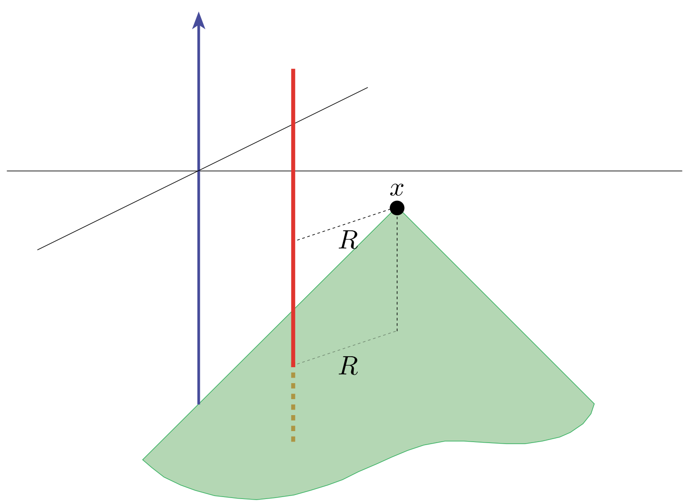

FIGURE 112: *Show in red is the worldline of a charged particle at rest (with respect to the inertial observer who drew the diagram). The distance from the field point x to the puncture point on the backward lightcone was seen to be R ... and so—as yet unbeknownst to the field point—it has remained.*

conforms nicely to our physical intuition. It is, therefore, the $\gamma(1-\beta_\parallel)$-term in (456.1) and the $(1-\beta_\parallel)$-term in (456.2) that demand "explanation" if we are to say that we "understand" (456). Now ... if θ is the angle subtended by $\boldsymbol{\beta}$

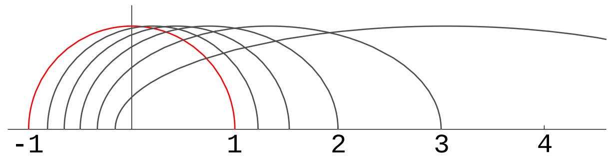

FIGURE 112: *Polar plots showing the θ-dependence of the Doppler factor* $\sqrt{1-\beta^2}/(1-\beta\cos\theta)$*, with* $\beta = 0, 0.2, 0.4, 0.6, 0.8, 0.95$.

and $\boldsymbol{R}$ we have

$$\frac{1}{\gamma(1-\beta_\parallel)} = \frac{1}{\gamma(1-\beta\cos\theta)} = \begin{cases} \sqrt{\dfrac{1+\beta}{1-\beta}} > 1 \text{ at } \theta = 0 \\ \qquad\quad = 1 \text{ at } \theta = \arccos\left[\dfrac{1-\sqrt{1-\beta^2}}{\beta}\right] \\ \qquad \dfrac{1}{\gamma} < 1 \text{ at } \theta = 90° \\ \sqrt{\dfrac{1-\beta}{1+\beta}} < 1 \text{ at } \theta = 180° \end{cases}$$

—results of which the preceding figure provides vivid graphic interpretations. The expressions $[(1+\beta)/(1-\beta)]^{\pm 1}$ are familiar (recall again PROBLEM 43) as the eigenvalues of $\mathbb{\Lambda}(\beta)$: they are found, morover, to be fundamental to the description of the *relativistic Doppler effect*,[270] so

$$\frac{1}{\gamma(1-\beta_\parallel)} \equiv \text{Doppler factor}$$

becomes[271] a natural terminology. Looking back again to (456.1), we see that the Doppler factor

- serves to *enhance* the value of A^μ_{R} if the source point is seen by the field point to be *approaching* at the moment of puncture:

$$0 \leqslant \theta_0 \leqslant \cos^{-1}\left[\tfrac{1-\sqrt{1-\beta^2}}{\beta}\right]_0$$

- serves in the contrary case to *diminish* the value of A^μ_{R}

. . . which is what one would expect if (see Figure 114) the lightcone possessed some small but finite "thickness," for in the former case the field point would then get a relatively "longer look" at the source point, and in the latter case a "briefer look." Note that it is not the Doppler factor itself but the

$$\text{truncated Doppler factor} \equiv \frac{1}{(1-\beta_\parallel)}$$

that stands in (456.2).

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[270] See ELECTRODYNAMICS (1972) page 239.

[271] Compare A. Sommerfeld, *Electrodynamics* (1952), page 250.

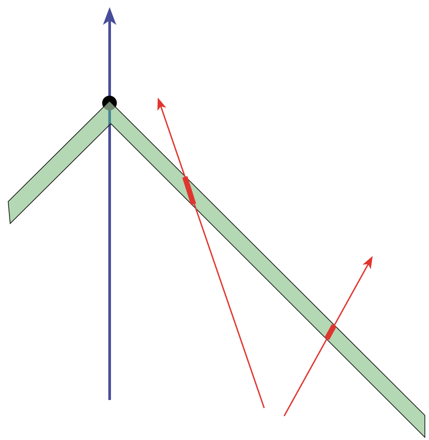

FIGURE 114: *If the lightcone had "thickness" then the presence of the Doppler factor in (456) could be understood qualitatively to result from the relatively "longer look" that the field point gets at approaching charges, the relatively "briefer look" at receding charges.*

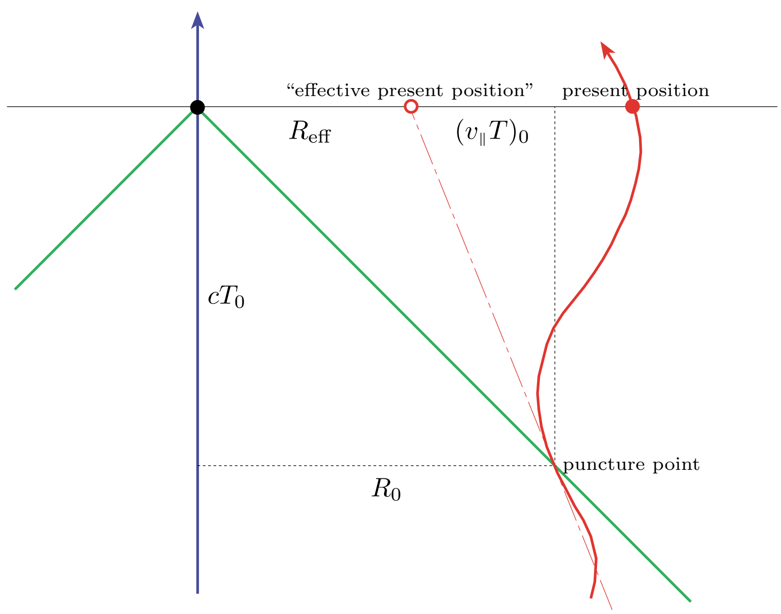

FIGURE 115: *Construction used to define the "effective present distance" from source to field point:*

$$R_{\text{eff}} = (R - v_{\parallel}T)_0 = (R - \beta_{\parallel}cT)_0 = (1 - \beta_{\parallel})_0 R_0$$

Some textbook writers make much of the curious fact that it is possible (see Figure 115) by linear extrapolation from the puncture point data to arrive at an "physical interpretation" of the expression $[(1-\beta_\parallel)R]_0$

$$[(1-\beta_\parallel)R]_0 = R_{\text{eff}} \equiv \begin{cases} \textit{present} \text{ distance from field point to charge if} \\ \text{the charge had moved uniformly/rectilinearly} \\ \text{since the moment of puncture} \end{cases}$$

and in this notation to cast (456.2) in the form

$$\begin{aligned} \varphi_{\text{R}}(x) &= \frac{e}{4\pi R_{\text{eff}}} \\ \boldsymbol{A}_{\text{R}}(x) &= \frac{e}{4\pi R_{\text{eff}}}\boldsymbol{\beta}_0 \end{aligned}$$

My own view is that the whole business, though memorably picturesque, should be dismissed as a *mere* curiosity . . . on grounds that it is too alien to the spirit of relativity—and to the letter of the principle of manifest Lorentz covariance—to be of "deep" significance. More worthy of attention, as will soon be demonstrated, is the fact that equations (456) admit[272] of the following *manifestly covariant* formulation

$$A^\mu_{\text{R}}(x) = \frac{e}{4\pi}\Big[\frac{u^\mu}{R_\alpha u^\alpha}\Big]_0 \tag{457}$$

where $$R^\mu \equiv x^\mu - x^\mu(\tau)$$

3. Field of a point source in arbitrary motion. What we want now to do is to evaluate

$$F^{\mu\nu}_{\text{R}}(x) = \partial^\mu A^\nu_{\text{R}}(x) - \partial^\nu A^\mu_{\text{R}}(x)$$

where $A^\mu_{\text{R}}(x)$ is given most conveniently by (457)

So <u>the physics of what follows is conceptually straightforward</u>. The point is worth keeping in mind, for the computational details are—like the final result—quite intricate.

Turning now, therefore, to the evaluation of

$$\partial^\mu A^\nu_{\text{R}}(x) = g^{\mu m}\frac{\partial}{\partial x^m}\Big\{\frac{e}{4\pi}\Big[\frac{u^\nu}{R_\alpha u^\alpha}\Big]_0\Big\}$$

. . . it is critically important to notice that (see the following figure) *variation of the field point x induces a variation of the proper time of puncture*; *i.e.*, that τ_0 is x-dependent: $\tau_0 = \tau_0(x)$. Formally,

$$\frac{\partial}{\partial x^m} = \frac{\boldsymbol{\partial}}{\boldsymbol{\partial} x^m} + \frac{\partial\,\tau}{\partial x^m}\frac{\partial}{\partial\,\tau}$$

where $\boldsymbol{\partial}_m$ senses *explicit* x-dependence and $(\partial_m\tau)\frac{\partial}{\partial\tau}$ senses *covert* x-dependence.

[272] PROBLEM 71.

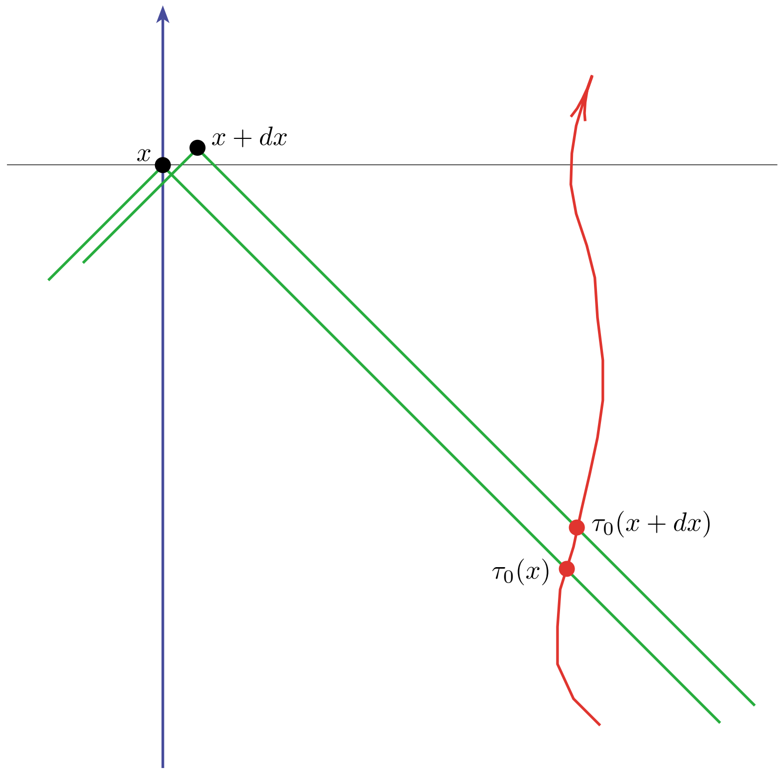

FIGURE 116: *Variation of the field point x typically entails variation also of the puncture point, and it is this circumstance that makes evaluation of the electromagnetic field components so intricate.*

Proceeding thus from

$$\partial^\mu A_{\mathrm{R}}^\nu(x) = \frac{e}{4\pi} g^{\mu m}\Big[\Big\{\frac{\boldsymbol{\partial}}{\boldsymbol{\partial} x^m} + \frac{\partial\tau}{\partial x^m}\frac{\partial}{\partial\tau}\Big\}\frac{u^\nu(\tau)}{R_\alpha(x,\tau)u^\alpha(\tau)}\Big]_0\Big\}$$

$$R_\alpha(x,\tau) \equiv x_\alpha - x_\alpha(\tau)$$

we are led by straightforward calculation to the following result:

$$\begin{aligned} &= \frac{e}{4\pi}\Big[\frac{1}{(R_\alpha u^\alpha)^2}\big(c^2 g^{\mu m}\frac{\partial\tau}{\partial x^m} - u^\mu\big)u^\nu\Big]_0 \\ &\quad + \frac{e}{4\pi}\Big[\frac{1}{(R_\alpha u^\alpha)}g^{\mu m}\frac{\partial\tau}{\partial x^m}\Big\{a^\nu - \frac{(R_\alpha a^\alpha)}{(R_\beta u^\beta)}u^\nu\Big\}\Big]_0 \end{aligned} \tag{458}$$

Here use has been made of $u^\alpha u_\alpha = c^2$ and also of

$$a^\mu \equiv \frac{du^\mu(\tau)}{d\tau} = \text{4-acceleration of the source particle}$$

Notational adjustments make this result easier to write, if not immediately easier to comprehend. Let r be the Lorentz-invariant length defined[270]

$$r \equiv \tfrac{1}{c} R_\alpha u^\alpha = \gamma(1-\beta_\parallel)R$$

and let w^μ be the dimensionless 4-vector defined

$$w^\mu \equiv c\partial^\mu\tau - \tfrac{1}{c}u^\mu$$

Then

$$\partial^\mu\tau = \frac{cw^\mu + u^\mu}{c^2}$$

Easily $\partial_\mu(R_\alpha R^\alpha) = 2\big\{R_\mu - (\partial_\mu\tau)(R_\alpha u^\alpha)\big\}$. From this and the fact that R^μ is (by definition of "puncture point") invariably null at the puncture point

$$\big[R_\alpha R^\alpha\big]_0 = 0, \quad \text{therefore} \quad \big[\partial_\mu(R_\alpha R^\alpha)\big]_0 = 0$$

it follows that

$$\big[\partial^\mu\tau\big]_0 = \big[R^\mu/(R_\alpha u^\alpha)\big]_0 = \tfrac{1}{c}\big[R^\mu/r\big]_0$$

from which

$$\begin{aligned} \big[u_\alpha w^\alpha\big]_0 &= 0 \\ \big[w_\alpha w^\alpha\big]_0 &= -1 \end{aligned}$$

$$\big[a_\alpha \partial^\alpha\tau\big]_0 = \left[\frac{R_\alpha a^\alpha}{R_\beta u^\beta}\right]_0 = \tfrac{1}{c}\big[a_\alpha w^\alpha\big]_0$$

follow as fairly immediate corollaries.[273] When we return with this information to (458) we obtain

$$\partial^\mu A^\nu_{\mathrm{R}}(x) = \frac{e}{4\pi}\left[\frac{1}{r^2}w^\mu b^\nu\right]_0 + \frac{e}{4\pi c^2}\left[\frac{1}{r}(w^\mu + b^\mu)\big(a^\nu - (aw)b^\nu\big)\right]_0$$

$$b \equiv \tfrac{1}{c}u = \gamma\begin{pmatrix} 1 \\ \boldsymbol{\beta}\end{pmatrix}$$

---

[273] For detailed proof see CLASSICAL RADIATION (1974), pages 523/4. But beware! I have now altered slightly the definitions of r and w^μ.

Consequently

$$
\begin{aligned}
F^{\mu\nu}_{\text{R}}(x) &\equiv \text{electromagnetic field at } x \text{ due to past source activity}\\
&= \frac{e}{4\pi}\Big[\frac{1}{r^2}(w^\mu b^\nu - w^\nu b^\mu)\Big]_0 \qquad (459)\\
&\quad + \frac{e}{4\pi c^2}\Big[\frac{1}{r}\big\{(b^\mu a^\nu - b^\nu a^\mu) + (w^\mu a^\nu - w^\nu a^\mu) - (aw)(w^\mu b^\nu - w^\nu b^\mu)\big\}\Big]_0\\
&= \boxed{\begin{array}{l}\text{acceleration-independent term} \sim 1/r^2,\\ \text{dominant near the worldline of the source}\end{array}}\\
&\qquad\qquad + \boxed{\begin{array}{l}\text{acceleration-dependent term} \sim 1/r,\\ \text{dominant far from the worldline of the source}\end{array}}\\
&= \boxed{\text{“velocity field”}} + \boxed{\text{“acceleration field”}}\\
&= \boxed{\text{“near field”}} + \boxed{\text{“far field”}}\\
&= \boxed{\text{generalized } \textbf{Coulomb} \text{ field}} + \boxed{\textbf{radiation} \text{ field}}
\end{aligned}
$$

This result is complicated (the physics is complicated!), but not “impossibly” complicated. By working in a variety of notations, from a variety of viewpoints, and in contact with a variety of special applications it is possible to obtain—ultimately—a fairly sharp feeling for the extraordinarily rich physical content of (459). As preparatory first steps toward that objective . . .

We note that, using results developed on the preceding page,

$$w^\mu = c\partial^\mu\tau - b^\mu$$

becomes

$$= \big[R^\mu/r - b^\mu\big]_0$$

which when spelled out in detail reads

$$\begin{pmatrix} w^0\\ \boldsymbol{w}\end{pmatrix} = \frac{1}{\gamma(1-\hat{\boldsymbol{R}}\cdot\boldsymbol{\beta})R}\begin{pmatrix} R\\ \boldsymbol{R}\end{pmatrix} - \gamma\begin{pmatrix} 1\\ \boldsymbol{\beta}\end{pmatrix}$$

with $\hat{\boldsymbol{R}} \equiv \boldsymbol{R}/R$. A little manipulation (use $\gamma^{-2} = 1 - \boldsymbol{\beta}\cdot\boldsymbol{\beta}$) brings this result to the form

$$= \frac{\gamma}{1-\hat{\boldsymbol{R}}\cdot\boldsymbol{\beta}}\begin{pmatrix} (\hat{\boldsymbol{R}}-\boldsymbol{\beta})\cdot\boldsymbol{\beta}\\ \hat{\boldsymbol{R}}-\boldsymbol{\beta}+\underbrace{(\hat{\boldsymbol{R}}\cdot\boldsymbol{\beta})\boldsymbol{\beta}-(\boldsymbol{\beta}\cdot\boldsymbol{\beta})\hat{\boldsymbol{R}}}_{=\,\boldsymbol{\beta}\times(\boldsymbol{\beta}\times\hat{\boldsymbol{R}})}\end{pmatrix} \qquad (460.1)$$

It follows similarly from (270) that

$$a = \begin{pmatrix} \dot{u}^0\\ \dot{\boldsymbol{u}}\end{pmatrix} = \gamma^4\begin{pmatrix} \boldsymbol{a}\cdot\boldsymbol{\beta}\\ \boldsymbol{a}+\boldsymbol{\beta}\times(\boldsymbol{\beta}\times\boldsymbol{a})\end{pmatrix} \qquad (460.2)$$

where $\boldsymbol{a} \equiv d\boldsymbol{v}/dt$.

To extract $\boldsymbol{E}(x)$ from (460) we have only (see again page 108) to set $\nu = 0$ and to let μ range on $\{1, 2, 3\}$:

$$\begin{aligned}\boldsymbol{E}(x) &= \frac{e}{4\pi}\Big[\frac{1}{r^2}(\boldsymbol{w}\,b^0 - w^0\boldsymbol{b})\Big]_0 \\ &\quad + \frac{e}{4\pi c^2}\Big[\frac{1}{r}\big\{(\boldsymbol{b}\,\dot{u}^0 - b^0\dot{\boldsymbol{u}}) + (\boldsymbol{w}\,\dot{u}^0 - w^0\dot{\boldsymbol{u}}) - (aw)(\boldsymbol{w}\,b^0 - w^0\boldsymbol{b})\big\}\Big]_0\end{aligned}$$

It follows readily from (460) that

$$\begin{aligned}\boldsymbol{w}\,b^0 - w^0\,\boldsymbol{b} &= \frac{1}{1 - \hat{\boldsymbol{R}}\cdot\boldsymbol{\beta}}(\hat{\boldsymbol{R}} - \boldsymbol{\beta}) \\ \boldsymbol{b}\,\dot{u}^0 - b^0\,\dot{\boldsymbol{u}} &= -\gamma^3\boldsymbol{a} \\ \boldsymbol{w}\,\dot{u}^0 - w^0\dot{\boldsymbol{u}} &= -\gamma^3\frac{1}{1 - \hat{\boldsymbol{R}}\cdot\boldsymbol{\beta}}\boldsymbol{\beta}\times\big(\boldsymbol{a}\times(\hat{\boldsymbol{R}} - \boldsymbol{\beta})\big) \\ (aw) &= -\gamma^3\frac{1}{1 - \hat{\boldsymbol{R}}\cdot\boldsymbol{\beta}}\Big\{(1-\beta^2)\,(\hat{\boldsymbol{R}}\cdot\boldsymbol{a}) - (1 - \hat{\boldsymbol{R}}\cdot\boldsymbol{\beta})(\boldsymbol{\beta}\cdot\boldsymbol{a})\Big\}\end{aligned}$$

so after some unilluminating manipulation we obtain

$$\begin{aligned}\boldsymbol{E}(x) &= \frac{e}{4\pi}\Big[\frac{1}{r^2}\frac{1}{(1 - \hat{\boldsymbol{R}}\cdot\boldsymbol{\beta})}(\hat{\boldsymbol{R}} - \boldsymbol{\beta})\Big]_0 \\ &\quad + \frac{e}{4\pi c^2}\Big[\frac{1}{r}\frac{\gamma}{(1 - \hat{\boldsymbol{R}}\cdot\boldsymbol{\beta})^2}\hat{\boldsymbol{R}}\times\big((\hat{\boldsymbol{R}} - \boldsymbol{\beta})\times\boldsymbol{a}\big)\Big]_0\end{aligned} \tag{461.1}$$

A similar[274] computation addressed to the evaluation of $\boldsymbol{B}(x)$ leads to a a result which can be expressed very simply/economically:

$$\boldsymbol{B}(x) = \Big[\hat{\boldsymbol{R}}\times\boldsymbol{E}(x)\Big]_0 \tag{461.2}$$

It should be noted that equations (459) and (461) describe precisely the same physics: they differ only notationally. And both are exact (no approximations). I remarked earlier, in connection with equations (456), that "the 'retarded evaluation' idea $\big[\quad\big]_0$ conforms nicely to our physical intuition," but must now admit that (461) contains many non-intuitive details: in this sense it is evidently *easier to think reliably about potentials (which are "spooks") than about fields (which are "real")*!

Notice also that if we insert the expressions that appear on the right sides of equations (461) into Lorentz' $\boldsymbol{F} = q(\boldsymbol{E} + \frac{1}{c}\boldsymbol{v}\times\boldsymbol{B})$ then we obtain, in effect, a description of the retarded position/velocity/acceleration-dependent action on one charge upon another—a description <u>free from any direct allusion to the field concept</u>! It was with the *complexity* of this and similar results in mind that I suggested (page 250) that life without fields "would . . . entail more cost than benefit."

[274] . . . and similarly tedious: generally speaking, one can expect tediousness to increase in proportion to how radically one departs—as here—from adherence to the principle of manifest covariance.

We have encountered evidence (pages 240, 297) of what might be called a "tendency toward $\boldsymbol{B} \perp \boldsymbol{E}$," but have been at pains to stress (page 332) that $\boldsymbol{B} \perp \boldsymbol{E}$ remains, nevertheless, an exceptional state of affairs. It is, in view of the latter fact, a little surprising to discover that $\boldsymbol{B} \perp \boldsymbol{E}$ does pertain—everywhere and exactly—to the field produced by a single point source in arbitrary motion. The key word here is "single," as I shall now demonstrate: write

- $\boldsymbol{E}$ and $\boldsymbol{B} = \hat{\boldsymbol{R}} \times \boldsymbol{E}$ to describe (at x) the field generated by e;
- $\boldsymbol{E}$ and $\boldsymbol{B} = \hat{\boldsymbol{R}} \times \boldsymbol{B}$ to describe the field generated by e.

Clearly $\boldsymbol{B} \perp \boldsymbol{E}$ and $\boldsymbol{B} \perp \boldsymbol{E}$. The question before us: "Is $(\boldsymbol{B} + \boldsymbol{B}) \perp (\boldsymbol{E} + \boldsymbol{E})$?" ... can be formulated "Does $(\hat{\boldsymbol{R}} \times \boldsymbol{E} + \hat{\boldsymbol{R}} \times \boldsymbol{E}) \cdot (\boldsymbol{E} + \boldsymbol{E}) = 0$?" and after a few elementary simplifications becomes "Does $(\boldsymbol{E} \times \boldsymbol{E}) \cdot (\hat{\boldsymbol{R}} - \hat{\boldsymbol{R}}) = 0$?" Pretty clearly, (461.1) carries no such implication unless restrictive conditions are imposed upon $\boldsymbol{\beta}$, $\boldsymbol{a}$, $\boldsymbol{\beta}$ and $\boldsymbol{a}$.[275, 276]

My plan now is to describe a (remarkably simple) physical interpretation of the acceleration-independent leading term in (461). This effort will motivate the introduction of certain diagramatic devices that serve to clarify the meaning also of the 2nd term. With our physical intuition thus sharpened, we will move in the next chapter to a discussion of the "radiative process."

4. Generalized Coulomb fields. The leading term in (459/461) provides an *exact* description of $\boldsymbol{E}(x)$ and $\boldsymbol{B}(x)$ if the source—as seen from x—is unaccelerated at the moment of puncture (*i.e.*, if $\boldsymbol{a}_0 = \mathbf{0}$), and it becomes universally exact (*i.e.*, exact for *all* fieldpoints x) for free sources (*i.e.*, for sources with rectilinear worldlines). Evidently

$$\boldsymbol{E} = \frac{e}{4\pi}\Big[\frac{1}{r^2}\frac{1}{(1-\hat{\boldsymbol{R}}\cdot\boldsymbol{\beta})}(\hat{\boldsymbol{R}}-\boldsymbol{\beta})\Big]_0 \tag{462.1}$$

$$r \equiv \gamma(1-\hat{\boldsymbol{R}}\cdot\boldsymbol{\beta})R \quad : \quad \text{see page 359}$$

$$\boldsymbol{B} = \Big[\hat{\boldsymbol{R}} \times \boldsymbol{E}(x)\Big]_0 \tag{462.2}$$

—which become "Coulombic" for sources seen to be at rest ($\boldsymbol{\beta} = \mathbf{0}$)—describe the *Lorentz transform of the electrostatic field generated by an unaccelerated*[277] *point charge*. They describe, in other words, our perception of the Coulomb field of a passing charge. Explicit proof—and interpretive commentary—is provided below.

We are, let us suppose, certifiably inertial. So also is O, whom we see to be drifting by with speed $\boldsymbol{\beta}$ (and whose habit it is to use red ink when writing

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[275] PROBLEM 72.

[276] PROBLEM 73.

[277] "Unaccelerated" is, we now see, redundant—implied already by the word "electro*static*." Readers may find it amusing/useful at this point to review the ideas developed in §2 of Chapter 1.

down his physical equations). It happens (let us assume) that O's frame is related irrotationally to ours; *i.e.*, by a pure boost $\mathbb{\Lambda}(\boldsymbol{\beta})$. Then (see again §5 in Chapter 2) the coordinates which he/we assign to a spacetime point stand in the relation

$$\left.\begin{aligned} t &= \gamma t + (\gamma/c^2)\,\boldsymbol{v}\cdot\boldsymbol{x} \\ \boldsymbol{x} &= \boldsymbol{x} + \big\{\gamma t + (\gamma-1)\,(\boldsymbol{v}\cdot\boldsymbol{x})/v^2\big\}\boldsymbol{v} \end{aligned}\right\} \tag{210.1}$$

which can be notated

$$\left.\begin{aligned} \begin{pmatrix} t \\ x_\parallel \end{pmatrix} &= \gamma\begin{pmatrix} 1 & v/c^2 \\ v & 1 \end{pmatrix}\begin{pmatrix} t \\ x_\parallel \end{pmatrix} \\ \boldsymbol{x}_\perp &= \boldsymbol{x}_\perp \end{aligned}\right\} \tag{210.2}$$

while the electric/magnetic fields which he/we assign to any given spacetime point stand in the relation

$$\left.\begin{aligned} \boldsymbol{E} &= (\boldsymbol{E}-\boldsymbol{\beta}\times\boldsymbol{B})_\parallel + \gamma(\boldsymbol{E}-\boldsymbol{\beta}\times\boldsymbol{B})_\perp \\ \boldsymbol{B} &= (\boldsymbol{B}+\boldsymbol{\beta}\times\boldsymbol{E})_\parallel + \gamma(\boldsymbol{B}+\boldsymbol{\beta}\times\boldsymbol{E})_\perp \end{aligned}\right\} \tag{263}$$

Let us suppose now that <u>O sees a charge e to be sitting at his origin</u>, and no magnetic field: $\boldsymbol{E} = \frac{e}{4\pi}R^{-2}\,\hat{\boldsymbol{x}}$ and $\boldsymbol{B} = \boldsymbol{0}$. The latter condition brings major simplifications to (263): we have

$$\boldsymbol{E} = \boldsymbol{E}_\parallel + \boldsymbol{E}_\perp \quad\text{with}\quad \begin{cases} \boldsymbol{E}_\parallel = \boldsymbol{E}_\parallel \\ \boldsymbol{E}_\perp = \gamma\boldsymbol{E}_\perp \end{cases}$$

$$\boldsymbol{B} = \boldsymbol{B}_\parallel + \boldsymbol{B}_\perp \quad\text{with}\quad \begin{cases} \boldsymbol{B}_\parallel = \boldsymbol{0} \\ \boldsymbol{B}_\perp = \gamma(\boldsymbol{\beta}\times\boldsymbol{E})_\perp = \boldsymbol{\beta}\times\boldsymbol{E} \end{cases}$$

which we see to be time-dependent (because we see the charge to be in motion). We use the notations introduced in Figure 117 to work out the detailed meaning of the preceding statements:

O sees a radial electric field:

$$\frac{E_\parallel}{E_\perp} = \frac{R_\parallel}{R_\perp}$$

But

$$\frac{R_\parallel}{R_\perp} = \frac{\gamma R_\parallel}{R_\perp} \quad : \quad \text{the } \|\text{-side of our space triangle is Lorentz contracted}$$

$$\frac{E_\parallel}{E_\perp} = \frac{E_\parallel}{\gamma^{-1}E_\perp} \quad : \quad \text{the } \perp\text{-component of our } \boldsymbol{E}\text{-field is Lorentz dilated}$$

so

$$\frac{E_\parallel}{E_\perp} = \frac{R_\parallel}{R_\perp} \quad : \quad \text{we } \textit{also} \text{ see a radial electric field}$$

But while O sees a spherical "pincushion," we (as will soon emerge) see a

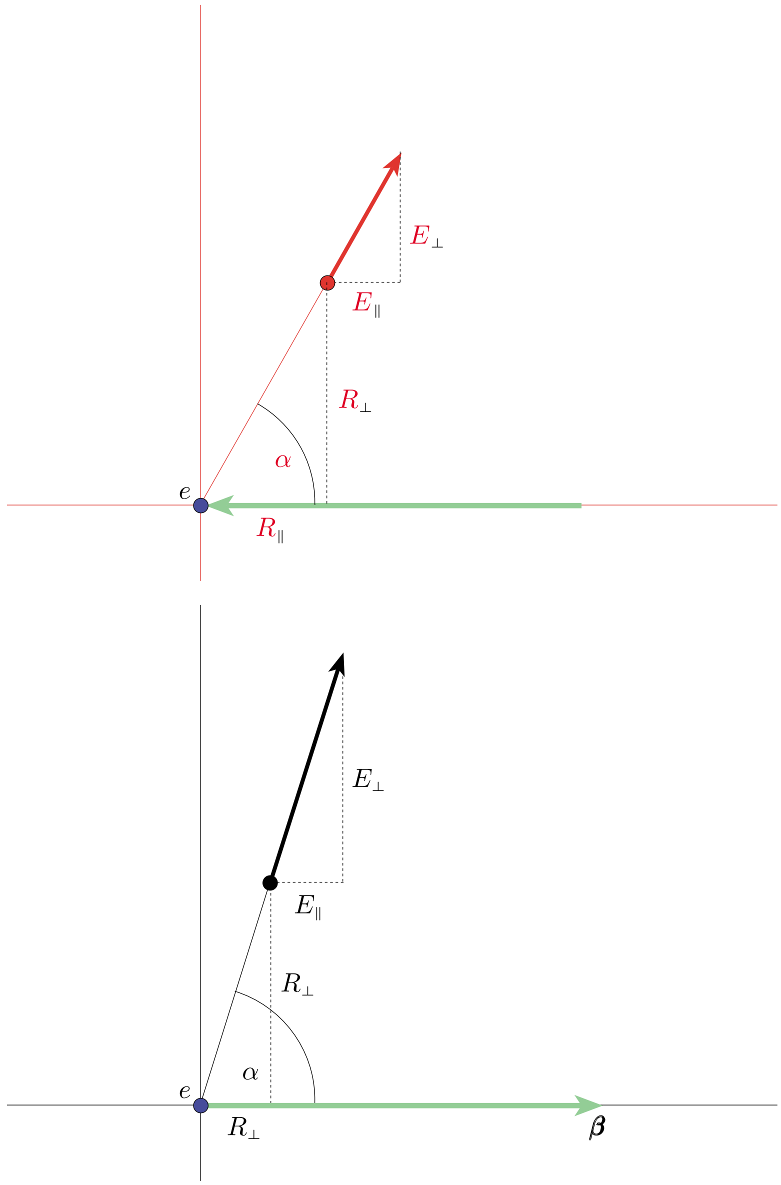

FIGURE 117: *Figures drawn on the space-plane that contains the charge •, the field-point in question, and the* $\boldsymbol{\beta}$*-vector with which the observer sees the other to be passing by. The upper figure defines the notation used by* O *to describe the Coulomb field of the charge sitting at his origin. The lower figure defines the notation we (in the text) use to describe our perception of that field.*

flattened pincushion. More precisely: O sees the field intensity to be given by

$$E = \frac{e}{4\pi R^2}\,, \ \text{independently of } \alpha$$

It follows, on the other hand, from the figure that

$$E = \sqrt{(E\cos\alpha)^2 + (\tfrac{1}{\gamma}E\sin\alpha)^2} = E\sqrt{\cos^2\alpha + \tfrac{1}{\gamma^2}\sin^2\alpha}$$

so

$$E = \frac{e}{4\pi R^2}\frac{1}{\sqrt{\cos^2\alpha + \frac{1}{\gamma^2}\sin^2\alpha}}$$

Similarly,

$$R = \sqrt{(\gamma R\cos\alpha)^2 + (R\sin\alpha)^2} = \gamma R\sqrt{\cos^2\alpha + \tfrac{1}{\gamma^2}\sin^2\alpha}$$

so

$$\begin{aligned} E &= \frac{e}{4\pi R^2}\frac{1}{\gamma^2\big(\cos^2\alpha + \frac{1}{\gamma^2}\sin^2\alpha\big)^{\frac{3}{2}}} \\ &= \frac{e}{4\pi R^2}\frac{1-\beta^2}{\big(1-\beta^2\sin^2\alpha\big)^{\frac{3}{2}}} \end{aligned} \tag{463.1}$$

which is to be inserted into

$$\boldsymbol{E} = E\hat{\boldsymbol{R}} \qquad \text{and} \qquad \boldsymbol{B} = \boldsymbol{\beta}\times\boldsymbol{E} \tag{463.2}$$

—the upshot of which is illustrated in Figures 118 & 119.

The results developed above make intuitive good sense, but do not much resemble (462). The discrepency is illusory, and arises from the circumstance that (462) is formulated in terms of the *retarded* position $\boldsymbol{R}_0$, while (463) involves the *present* position $\boldsymbol{R}$. Working from Figures 120 & 121 we have

$$\boldsymbol{R} = \boldsymbol{R}_0 - R_0\boldsymbol{\beta}$$

which is readily seen[278] to entail

$$R = R_0\sqrt{1 - 2\hat{\boldsymbol{R}}_0\!\cdot\!\boldsymbol{\beta} + \beta^2} = R_0\sqrt{1 - 2\beta\cos\theta + \beta^2}$$

Also[278]

$$\sin^2\alpha = (R_0/R)^2\sin^2\theta = \frac{1-\cos^2\theta}{1-2\beta\cos\theta+\beta^2}$$

and with this information—together with the observation that

$$\boldsymbol{\beta}\times\boldsymbol{E} = \frac{\boldsymbol{R}_0 - \boldsymbol{R}}{R_0}\times E\hat{\boldsymbol{R}} = \hat{\boldsymbol{R}}_0\times\boldsymbol{E}$$

—it is an easy matter to recover (462) from (463).[278]

[278] PROBLEM 74.

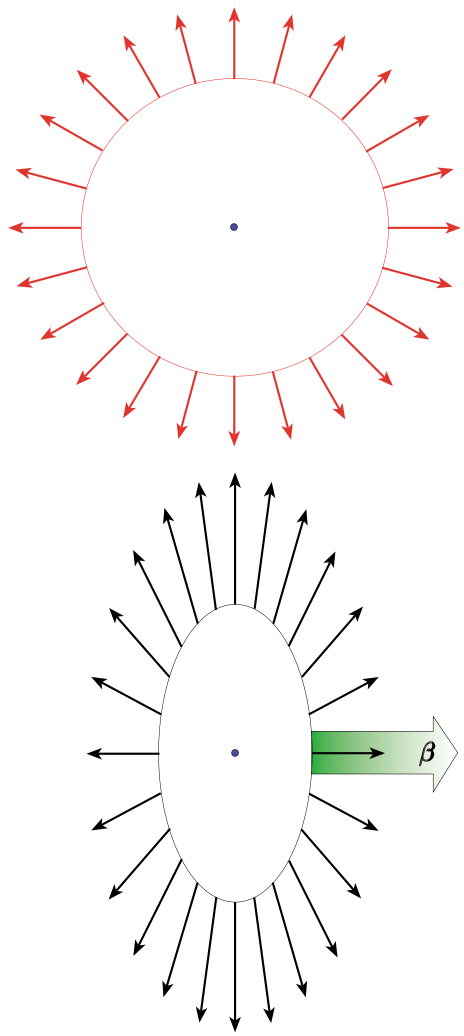

FIGURE 118: *Above: cross section of the "spherical pincushion" that O uses to represent the Coulomb field of a charge • which he sees to be at rest. We see the charge to be in uniform rectilinear motion. The "flattened pincushion" in the lower figure (axially symmetric about the $\boldsymbol{\beta}$-vector) describes our perception of that same electric field. Additionally, we see a solinoidal magnetic field given by*

$$\boldsymbol{B} = \boldsymbol{\beta} \times \boldsymbol{E}$$

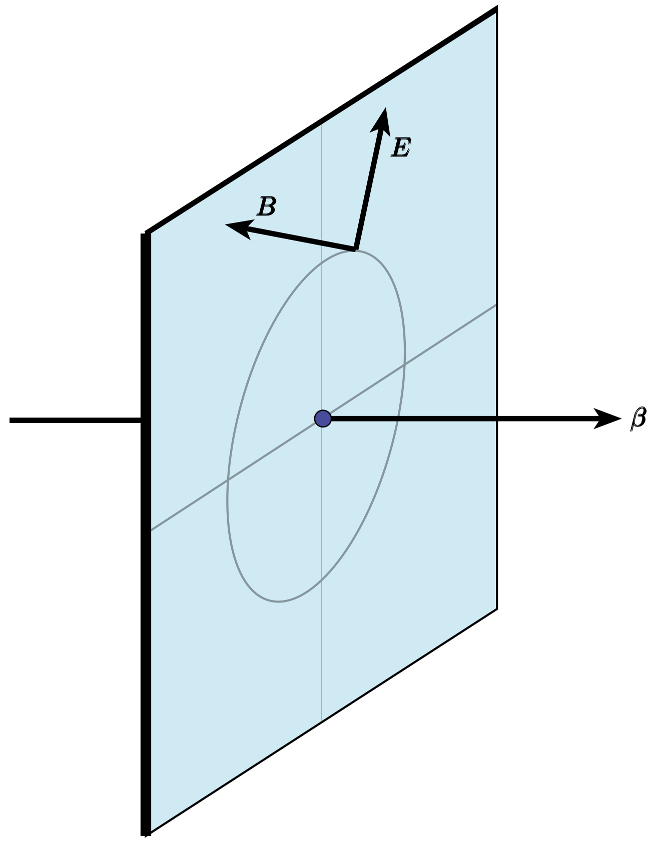

FIGURE 119: *Ultrarelativistic version of the preceding figure, showing also the solenoidal magnetic field. The "pincushion" has become a "pancake:" the field of the rapidly-moving charge is seen to be very nearly confined to a plane, outside of which it nearly vanishes, but within which it has become very strong.*

A **curious cautionary remark** is now in order. We have several times spoken casually/informally of the Coulomb fields "seen" by O and by us. Of course, one does not literally "see" a Coulomb field as one might see/photograph a passing *object* (a *literal* pincushion). The photographic appearance of an object (assume infinitely fast film and shutter) depends actually upon whether it is continuously/intermittently illuminate/self-luminous: the remarks which follow are (for simplicity) specific to continuously self-luminous objects. An object traces a "worldtube" in spacetime. The worldtubes of objects in motion (relative to us) are Lorentz-contracted in the $\boldsymbol{\beta}$-direction. What we see/ photograph is the *intersection of the Lorentz-contracted worldtube with the lightcone that extends into the past from the eye/camera*. The point—once stated—is obvious, but its surprising consequences passed unnoticed until 1959,

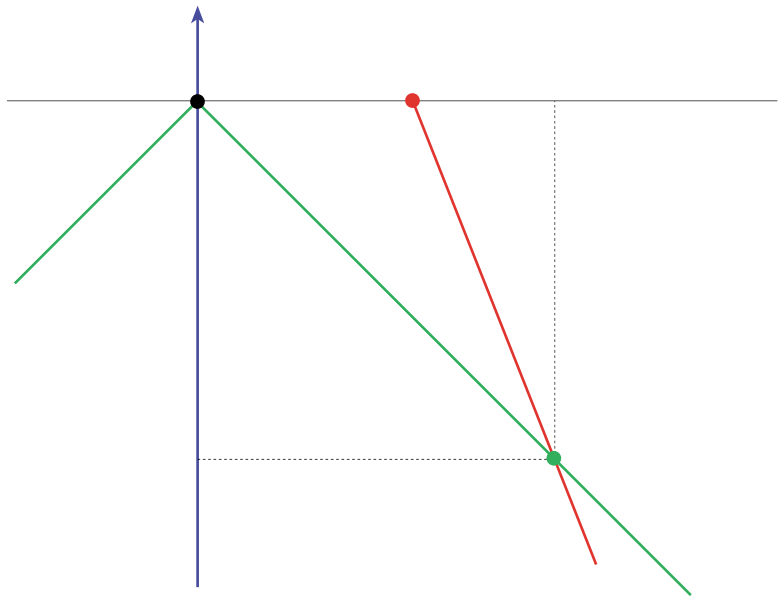

FIGURE 120: *Variant of Figure 115 in which the motion of the charge is not just "pretend unaccelerated" but really unaccelerated. In this spacetime diagram the chosen field point is marked •, the puncture point visible from • is marked •, while • marks the present position of the charge.*

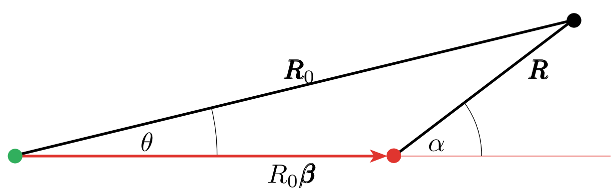

FIGURE 121: *Representation of the spatial relationship among the points •, • and •, which lie necessarily in a plane. A signal proceeds • → • with speed c in time $T_0 = R_0/c$, during which time the charge has advanced a distance $vT_0 = \beta R_0$ in the direction $\hat{\boldsymbol{\beta}}$. This little argument accounts for the lable that has been assigned to the red base of the triangle (i.e., to the charge displacement vector).*

when they occurred independently to J. Terrell and R. Penrose. For discussion, computer-generated figures and detailed references see (for example) G. D. Scott & H. J. van Driel, "The geometrical appearance of large objects moving at relativistic speeds," AJP **33**, 534 (1965); N. C. McGill, "The apparent shape of rapidly moving objects in special relativity," Contemp. Phys. **9**, 33 (1968); Ya. A. Smorodinskiĭ & V. A. Ugarov, "Two paradoxes of the special theory of relativity," Sov. Phys. Uspekhi **15**, 340 (1972). I am sure a search would turn up also many more recent sources.

It is important to appreciate that our principal results—equations (462) and (463)—might alternatively have been derived by a *potential-theoretic line of argument*, as sketched below: O, who sees the charge e to be at rest, draws upon (363) to write

$$\begin{aligned}\boldsymbol{E} &= -\boldsymbol{\nabla}\varphi - \tfrac{1}{c}\tfrac{\partial}{\partial t}\boldsymbol{A}\\ \boldsymbol{B} &= \boldsymbol{\nabla}\times\boldsymbol{A}\end{aligned}$$

where

$$A = \begin{pmatrix}\varphi\\ \boldsymbol{A}\end{pmatrix} \equiv \begin{pmatrix}e/4\pi R\\ \boldsymbol{0}\end{pmatrix}$$

entails

$$\boldsymbol{E} = -\boldsymbol{\nabla}\varphi = (e/4\pi R^2)\hat{\boldsymbol{R}} \qquad\text{and}\qquad \boldsymbol{B} = \boldsymbol{0}$$

O sees $\boldsymbol{E}$ to be normal to the equipotentials (surfaces of constant φ), which are themselves spherical (see again the upper part of Figure 118). On the other hand we—who see the charge to be in uniform motion—write

$$A = \mathbb{A}(-\boldsymbol{\beta})A = \gamma\phi\begin{pmatrix}1\\ \boldsymbol{\beta}\end{pmatrix}$$

with

$$\phi(x) = \varphi(\boldsymbol{x}(\boldsymbol{x},t)) = \frac{e}{4\pi\sqrt{\gamma^2(\boldsymbol{x}_\parallel - \boldsymbol{v}t)\cdot(\boldsymbol{x}_\parallel - \boldsymbol{v}t) + \boldsymbol{x}_\perp\cdot\boldsymbol{x}_\perp}}$$

and (drawing similarly upon (363)) obtain

$$\begin{aligned}\boldsymbol{E} &= -\big\{\boldsymbol{\nabla} + \boldsymbol{\beta}\tfrac{1}{c}\tfrac{\partial}{\partial t}\big\}\varphi \quad\text{with}\quad \varphi \equiv \gamma\phi\\ \boldsymbol{B} &= -\big\{\boldsymbol{\beta}\times\boldsymbol{\nabla}\big\}\varphi\end{aligned}$$

from which (462/463) can (with labor) be recovered. Note that *we* consider the equipotentials to be *ellipsoidal* (see again the lower part of Figure 118), and that the $\boldsymbol{\beta}\frac{1}{c}\frac{\partial}{\partial t}\varphi$-term causes the $\boldsymbol{E}$-field to be <u>no longer normal</u> to the equipotentials.

Useful geometrical insight into analytical results such as those developed above (and in the next chapter) can be obtained if one looks to the structure of the so-called "equiphase surfaces" which (see Figure 122) are inscribed on timeslices by lightcones projected forward from source points. The points which collectively comprise an equiphase surface "share a puncture point," but in the general case (*i.e.*, except when the source is seen to be momentarily at rest) share little else. To the experienced eye they do, however, indicate at least the qualitative essentials of field structure . . . as will emerge.

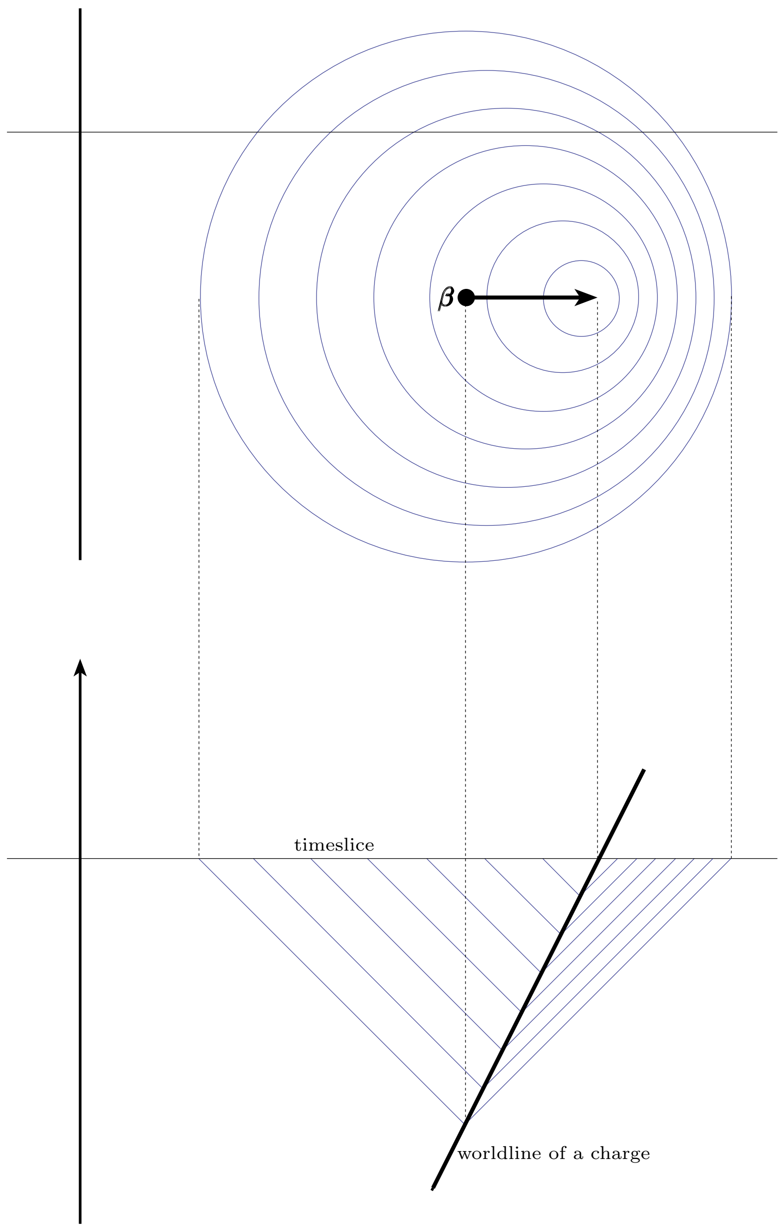

FIGURE 122: *Above: "equiphase surfaces" inscribed on a timeslice by (in this instance) a solitary charge in uniform motion (lower spacetime diagram). More complicated variants of the figure will be encountered in the next chapter.*